Math 251 Midterm 2

Name: ____________________

This exam has 9 questions, for a total of 100 points + 10 bonus points.

Please answer each question in the space provided. Please write **full solutions**, not just answers. Cross out anything the grader should ignore and circle or box the final answer.

Question	Points	Score
1	10	
2	14	
3	14	
4	10	
5	10	
6	10	
7	12	
8	20	
Total:	100	

Question	Bonus Points	Score
Bonus Question 1	10	
Total:	10	

Question 1. (10 pts)

Consider the function

$$f(x, y, z) = x^2 + z \sin y$$

(a) Find the gradient of the function $f(x, y, z)$

Solution:

$$\begin{aligned} \nabla f &= \langle f_x, f_y, f_z \rangle \\ &= \langle 2x, \quad z \cos y, \quad \sin y \rangle \end{aligned}$$

(b) Find the maximum rate of change of $f(x, y, z)$ at the point $(1, \pi, 1)$. In which direction does this maximum rate of change occur?

Solution: The maximum rate of change at $(1, \pi, 1)$ is

$$\|\nabla f(1, \pi, 1)\| = \|\langle 2, -1, 0 \rangle\| = \sqrt{5}$$

It occurs in the direction of the vector

$$\left\langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, 0 \right\rangle$$

Question 2. (14 pts)

Given

$$f(x, y) = 2x^2 + y^2 - xy^2 + 100$$

Determine all local maximum, minimum and saddle points.

Solution: First, find all critical points. We need to solve

$$\begin{cases} f_x = 4x - y^2 = 0 \\ f_y = 2y - 2xy = 0 \end{cases}$$

The second equation is $2(1-x)y = 0$. **Case 1:** $x = 1$, then from the first equation, we see

$$y^2 = 4$$

So $(1, \pm 2)$ are critical points.

case 2: $y = 0$, then $x = 0$. So $(0, 0)$ is a critical point.

Use the second derivative test,

$$f_{xx} = 4$$

$$f_{yy} = 2 - 2x$$

$$f_{xy} = -2y$$

So

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 8(1-x) - 4y^2$$

(1) At $(0, 0)$, $D(0, 0) = 8$ and $f_{xx}(0, 0) = 4$, so $(0, 0)$ is a local minimum.

(2) At $(1, \pm 2)$, $D(1, \pm 2) = -16 < 0$, so $(1, \pm 2)$ are saddle points.

Question 3. (14 pts)

Use the Lagrange multiplier method to find the absolute extreme values of the function

$$f(x, y) = 3x + y$$

with the constraint $x^2 + y^2 = 10$.

Solution: The constraint is

$$g(x, y) = x^2 + y^2 - 10 = 0$$

Apply the Lagrange multiplier method,

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x, y) = 0 \end{cases}$$

That is,

$$\begin{cases} 3 = \lambda(2x) \\ 1 = \lambda(2y) \\ x^2 + y^2 - 10 = 0 \end{cases}$$

Using the first two equations, we get

$$x = 3y.$$

Plug this back into the equation

$$x^2 + y^2 - 10 = 0$$

we get $y = \pm 1$. So we have two points

$$(3, 1) \text{ and } (-3, -1)$$

So the absolute max is $f(3, 1) = 10$, and the absolute min is $f(-3, -1) = -10$.

Question 4. (10 pts)

Rewrite (but do not evaluate)

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{8-x^2-y^2} (x+zy)\ dzdxdy$$

in cylindrical coordinates.

Solution: From the bounds, we can see that the region is exactly everything between the cone $z=\sqrt{x^2+y^2}$ and the upper hemisphere $z=\sqrt{2-x^2-y^2}$.

$$\int_{-\pi/2}^{\pi/2}\int_{0}^{2}\int_{r}^{8-r^2} (r\cos\theta + zr\sin\theta)r\ dzdrd\theta$$

Question 5. (10 pts)

Evaluate the following integral by switching the order of integration.

$$\int_0^1 \int_x^1 e^{x/y}\, dydx$$

Solution:

$$\int_0^1 \int_x^1 e^{x/y}\, dydx$$

$$= \int_0^1 \int_0^y e^{x/y}\, dxdy$$

use substitution $u = x/y$.

$$= \int_0^1 ye^{x/y}\Big|_{x=0}^{x=y} dy = \int_0^1 (e-1)ydy$$

$$= \frac{e-1}{2}$$

Question 6. (10 pts)

Rewrite (but do not evaluate)

$$\int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} (x^2+y^2+x)dxdy$$

in polar coordinates.

Solution:

$$\int_0^{\pi} \int_0^{2\sin\theta} (r^2 + r\cos\theta)r\ drd\theta$$

Question 7. (12 pts)

Find the area of the region inside the circle $x^2 + y^2 = 4x$ and outside the circle $x^2 + y^2 = 2x$.

Solution: use polar coordinates. We shall write the given equations in polar coordinates.

$$x^2 + y^2 = 4x \quad \Rightarrow \quad r^2 = 4r\cos\theta, \text{ i.e. } r = 4\sin\theta$$

$$x^2 + y^2 = 2x \quad \Rightarrow \quad r = 2\cos\theta$$

$$\begin{aligned}
\text{Area } &= \iint_D dA \\
&= \int_{\pi/2}^{-\pi/2} \int_{2\cos\theta}^{4\cos\theta} r dr d\theta \\
&= \int_{\pi/2}^{-\pi/2} \frac{r^2}{2}\Big|_{2\cos\theta}^{4\cos\theta} dr d\theta \\
&= \int_{\pi/2}^{-\pi/2} 6\cos^2\theta d\theta = \int_{\pi/2}^{-\pi/2} 3(1+\cos(2\theta))d\theta \\
&= \cdots = 3\pi
\end{aligned}$$

Question 8. (20 pts)

E is the solid that is below the upper half of the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{3x^2 + 3y^2}$.

(a) Write the volume of E as a triple integral in xyz-coordinates.

Solution:

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{4-x^2-y^2}} \; dzdxdy$$

(b) Write the volume of E as a triple integral in cylindrical coordinates.

Solution:

$$\int_{0}^{2\pi} \int_{0}^{1} \int_{\sqrt{3r^2}}^{\sqrt{4-r^2}} r \; dzdrd\theta$$

(c) Write the volume of E as a triple integral in spherical coordinates.

Solution:

$$\int_0^{2\pi} \int_0^{\pi/6} \int_0^2 \rho^2 \sin\varphi \; d\rho d\varphi d\theta$$

We get the bounds for φ by writing the cone $z = \sqrt{3x^2 + 3y^2}$ in terms of spherical coordinates

$$\rho \cos\varphi = \sqrt{3\rho^2 \sin^2\varphi}$$

so

$$\tan\varphi = \frac{\sqrt{3}}{3} \Rightarrow \varphi = \frac{\pi}{6}$$

(d) Use one of your answers from part (a), (b) and (c) to calculate the volume of E.

Solution: Use spherical coordinates

$$\int_0^{2\pi} \int_0^{\pi/6} \int_0^2 \rho^2 \sin\varphi \; d\rho d\varphi d\theta$$
$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{\rho^3}{3}\Big|_{\rho=0}^{\rho=2} \sin\varphi \; d\varphi d\theta$$
$$= \frac{16\pi}{3}\left(1 - \frac{\sqrt{3}}{2}\right)$$

Bonus Question 1. (10 pts)

E is a solid that is below the sphere $x^2+y^2+z^2=2z$ and above the cone $z=\sqrt{x^2+y^2}$.

(a) Express its volume by a triple integral in xyz-coordinates.

Solution: To get the upper bound, we need to use the equation $x^2+y^2+z^2=2z$ to write z in terms of x and y. This amounts to solving a quadratic equation.

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{\sqrt{x^2+y^2}}^{1+\sqrt{1-x^2-y^2}} \quad dzdxdy$$

(b) Use an appropriate coordinate system to evaluate the volume of E.

Solution:

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\varphi}\rho^2\sin\varphi\ d\rho d\varphi d\theta=\cdots=\pi$$

We get the upper bounds for φ by writing $x^2+y^2+z^2=2z$ in terms of spherical coordinates

$$\rho^2=2\rho\cos\varphi\Rightarrow\rho=2\cos\varphi$$